Solved Problems of Mechanics
Chapter-9 Vertical Circular Motion

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Problem 9.1: A particle slides from rest at a depth $R/2$ below the highest point down the outside of a smooth sphere of radius R. Prove that it leaves the sphere at a height $R/3$ above the Center. (You can make reasonable assumptions.)

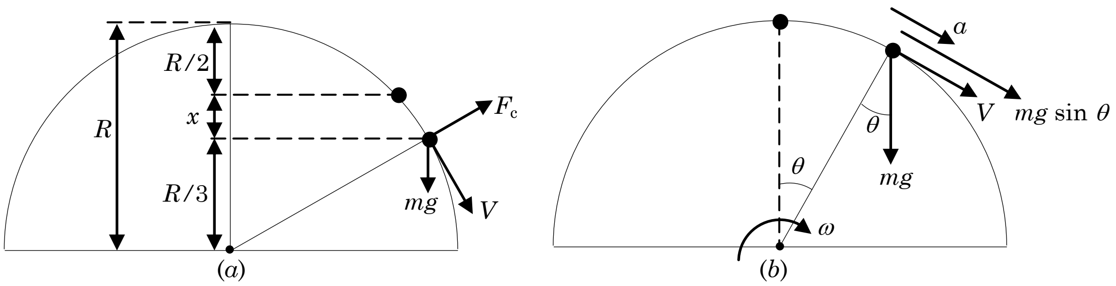

Diagram 9.1 Problem 9.1

Solution: Refer to Diagram 9.1a.

$$x = R - (R + R/3) = R/6$$

Calculating time: A particle of mass m is released from the top of the hemisphere with almost negligible velocity. Thus here we describe the motion of the particle in terms of time.

Refer to Diagram 9.1b,

Linear acceleration $\quad a = g \sin \theta,$

Angular acceleration, $\quad \alpha = a/R = (g \sin \theta)/R,$

Instantaneous velocity, $\quad \omega = V/R,$

From energy balance, $\quad mg\,(R - R\cos\theta) = \dfrac{1}{2} m V^2$

or, $$V = \sqrt{2g\,(R - R\cos\theta)}$$

Thus, $$\omega = \frac{V}{R} = \frac{\sqrt{2g\,(R - R\cos\theta)}}{R}$$

or, $$\omega = \frac{d\theta}{dt} = \frac{\sqrt{2g}}{R}\sqrt{(1-\cos\theta)}$$

or, $$\int_0^\theta \frac{d\theta}{\sqrt{(1-\cos\theta)}} = \frac{\sqrt{2g}}{R}\int_0^t dt$$

On integration we get the angular displacement, and then we get the height.

Problem 9.2: A particle of mass m attached to a massless rigid describes a vertical circular motion The speeds at the lowest and the highest positions are $3U$ and U respectively, prove that the tension in the rod when its angle with the downward vertical is θ, is equal to $mg(3\cos\theta + 2.5)$.

Solution: Refer to Diagram 9.2.

From energy balance,

$$-2mgl = \frac{1}{2}mU^2 - \frac{1}{2}m(3U)^2$$

$$U = \sqrt{gl/2} \qquad [i]$$

Again from energy balance,

$$-mg\,(l - l\cos\theta) = \frac{1}{2}mV^2 - \frac{1}{2}m(3U)^2$$

$$mV^2 = 9mU^2 - 2mg\,(l - l\cos\theta) \qquad [ii]$$

Force balance equation,

$$T = mg\cos\theta + F_c = mg\cos\theta + \frac{mV^2}{l} \qquad [iii]$$

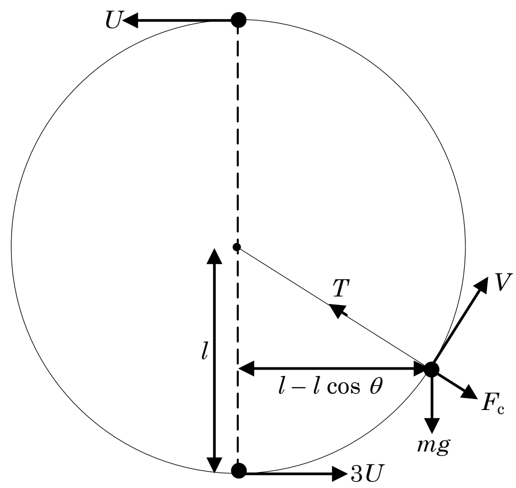

Diagram 9.2 Problem 9.2

On solving Eqns. (i), (ii), and (iii) we have

$$T = mg\cos\theta + F_c$$

$$= mg\cos\theta + \frac{9mU^2 - 2mg\,(l - l\cos\theta)}{l}$$

On solving we have

$$T = 3mg\cos\theta + 2.5\,mg$$

Problem 9.3: A particle is connected to rigid rod in a zero gravity region. Initially the particle is in rest at the position shown in Diagram 9.3a. A constant force F has been applied on the particle. Find out the maximum velocity of the particle and also find out the acceleration at that position. The initial angle made by the force vector with the radius is θ.

Solution: In this problem the constant force F placed the same roll which mg would have placed in a uniform gravitational field. The velocity of the particle will be maximum at point A [Diagram 9.3b],

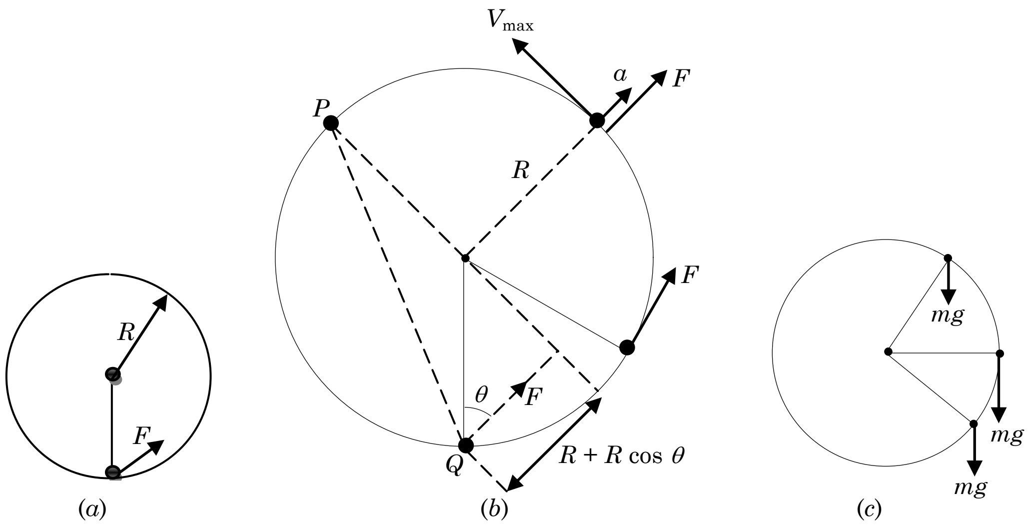

Diagram 9.3 Problem 9.3

From torque balance,

$$F(R + R\cos\theta) = \frac{mV^2}{2}$$

At the point A the tangential acceleration of the particle is zero, and we have only radial acceleration which is V_{max}^2/R.

Thus, $$F(R + R\cos\theta) = \frac{mV_{max}^2}{2}$$

or, $$V_{max} = \sqrt{\frac{2F(R + R\cos\theta)}{m}}$$

On measure the arc PQ the particle will perform oscillation.

Problem 9.4: A heavy particle of mass m, oscillates through an angle 180° on the inside of a smooth circular hoop of radius R fixed in a vertical plane. Prove that the normal reaction on the hoop at any point is $3mV^2/2R$, where V is the velocity at that point.

Solution: Refer to Diagram 9.4.

From energy balance,

$$mgR\cos\theta = \frac{1}{2}mV^2$$

From force balance,

$$N = mg\cos\theta + \frac{mV^2}{R}$$

or, $$N = \frac{mV^2}{R} + \frac{mV^2}{2R} = \frac{3mV^2}{2R}$$

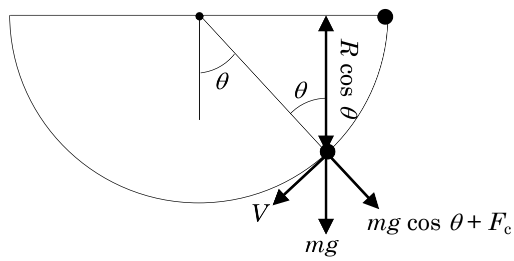

Diagram 9.4 Problem 9.4

Problem 9.5: Inside the spherical tube shown, a small particle of mass m is oscillating through an angle 180°. When the particle is at its lowest position, the tube is given a constant horizontal acceleration of $2g$. Find out the angle on which the mass will oscillate. Radius is R. No friction. The inner and the outer radius are almost equal [Diagram 9.5a].

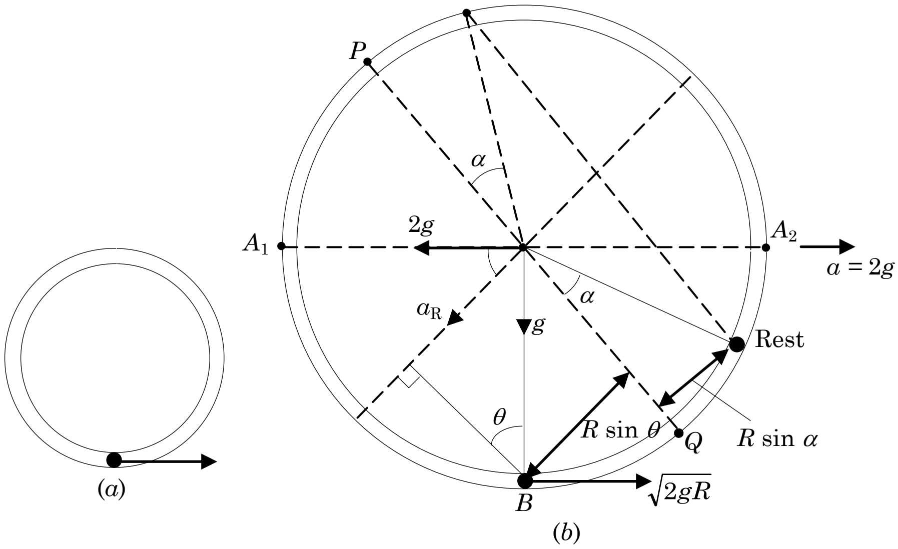

Diagram 9.5 Problem 9.5

Solution: We are analyzing the motion of particle with respect to tube. The distance travelled by the particle opposite to the resultant acceleration vector [Diagram 9.5b] is R $\sin\theta + R\sin\alpha$.

Resultant acceleration,

$$a_R = \sqrt{(2g)^2 + g^2} = g\sqrt{5}$$

From energy balance,

$$-ma_R(R\sin\theta + R\sin\alpha) = -\frac{1}{2}mV^2$$

or, $$g\sqrt{5}R(\sin\theta + \sin\alpha) = \frac{2gR}{2}$$

or, $$\sin\theta + \sin\alpha = \frac{1}{\sqrt{5}}$$

Since, $a_R \sin\theta = g$

Thus, $$\sin\theta = \frac{1}{\sqrt{5}}, \text{ and } \sin\alpha = 0.$$

The particle come to a state of rest with respect to a tube at a point Q, its oscillates on 180° but the maximum position has shifted.

Problem 9.6: A mass m hangs at one end of a string of length l, the other end is fixed. The mass is given a horizontal velocity $[7gl/2]^{1/2}$. Show that the tension in the string becomes zero when the string makes an angle 60° with upward vertical. Also show that tension in the string at that position is $3mg$.

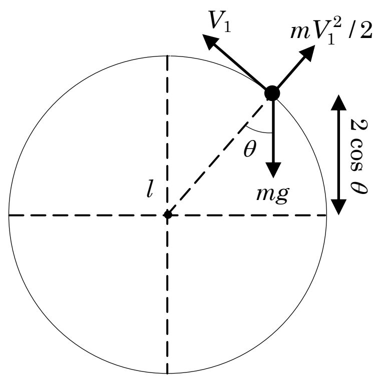

Diagram 9.6 Problem 9.6

Solution: Refer to Diagram 9.6.

From energy balance,

$$-mg(l + l\cos\theta) = \frac{1}{2}mV_1^2 - \frac{1}{2}mV^2$$

From force balance, $mg\cos\theta = \dfrac{mV_1^2}{l}$

Therefore, $-2mgl(1+\cos\theta) = mg\cos\theta + \dfrac{m}{l}\dfrac{2gl}{2}$

On solving we have, $\cos\theta = 1/2$, or, $\theta = 60°$.

Problem 9.7: A particle hanging from a fixed point by a string of length l, is projected horizontally with speed $[gl]^{1/2}$. Find the speed of the particle and the inclination of the string to the vertical at the instant of motion when the tension in the string is equal to its weight.

Solution: Refer to Diagram 9.7. let $V = \sqrt{gl}.V$

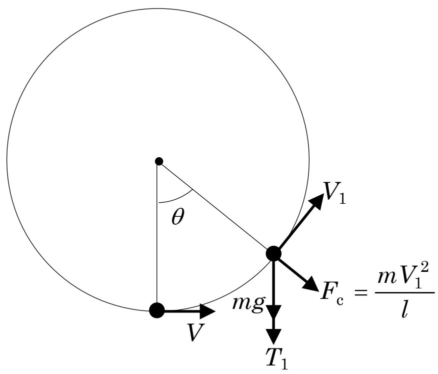

Diagram 9.7 Problem 9.7

From energy balance,

$$-mg(l - l\cos\theta) = \frac{1}{2}mV_1^2 - \frac{1}{2}mV^2$$

From force balance,

$$T = mg\cos\theta + \frac{mV_1^2}{l}$$

or, $$mg(1-\cos\theta) = \frac{mV_1^2}{l} \quad \{\therefore\ T = mg\}$$

Therefore, $-mgl(1-\cos\theta) = \dfrac{mV_1^2}{2} - \dfrac{mV^2}{2}$

On solving we have, $V_1 = \sqrt{gl/3}$.

Problem 9.8: A particle connected to a string of length l, has velocity $[6gl]$ at the lowest position. The tension at the highest position is T_1. Now, nail is fixed at a distance x from the center on the horizontal diameter. The radius of the new circle is $l - x$. At the highest point of the new circle let the tension be T_2.
If $T_2/T_1 = 3$ then find out the value of x.

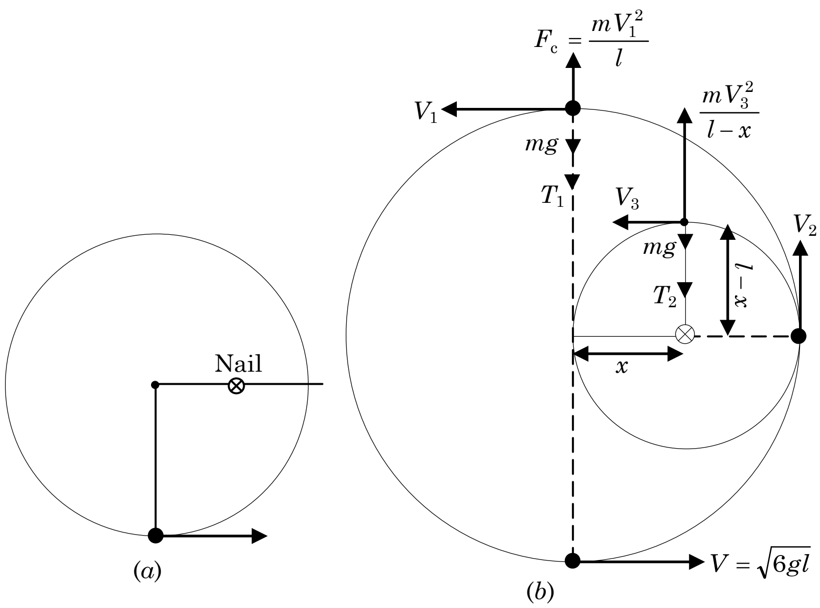

Diagram 9.8 Problem 9.8

Solution: Refer to Diagram 9.8.
From energy balance,

$$-2mgl = \frac{1}{2}mV_1^2 - \frac{1}{2}mV^2$$

From force balance,

$$T_1 + mg = \frac{mV_1^2}{l}$$

Therefore, $$-2mgl = \frac{mV_1^2}{2} - 6mgl$$

or, $$V_1 = \sqrt{2gl}\text{ , and } T_1 = mg.$$

Again energy balance from the lowest point from to the highest point of the new circle.

$$-2mg\,(l + l - x) = \frac{1}{2}mV_3^2 - \frac{1}{2}mV^2$$

or, $$mV_3^2 = 2mgl + 2mgx$$

Again from force balance,

$$T_2 = \frac{mV_3^2}{l - x} - mg$$

Therefore, $$T_2 = \frac{2mgl + 2mgx}{l - x} - mg$$

or, $$T_2 = \frac{2mg\,(l + x)}{l - x} - mg$$

According to problem, $T_2/T_1 = 3$, thus,

$$3 = \frac{mg\,l + 3mgx}{l - x}$$

On solving, we get the value of *x*.

Problem 9.9: A particle slides from the top down the outside smooth surface of a fixed sphere of radius *R*. Find the initial horizontal velocity that must be given so that the particle leaves the surface at a point, whose vertical height above the center of the sphere is $3R/4$.

Solution: Refer to Diagram 9.9.
From energy balance,

$$mg\left(R - \frac{3}{4}R\right) = \frac{1}{2}mV_2^2 - \frac{1}{2}mV_1^2$$

From force balance,

$$mg\cos\theta = \frac{mV_2^2}{R}$$

Therefore, $$-mg\frac{R}{4} + mgR\cos\theta = mV_1^2$$

Let $\theta = 0°$, because we find the velocity at initial position,

or, $$V_1 = \sqrt{gR/2}\,.$$

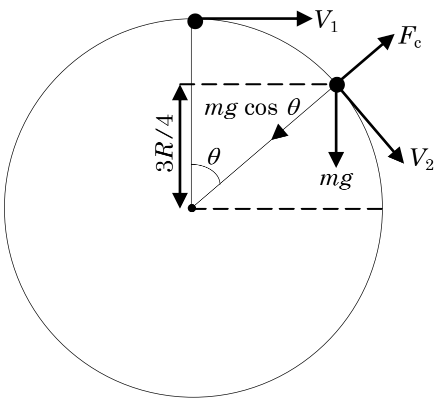

Diagram 9.9 Problem 9.9

Problem 9.10: A small body is placed on the top of a smooth sphere of radius R. The sphere is given a constant horizontal acceleration A. The body starts sliding on the sphere. When θ is the angle between the radius vector and the vertical, the particle looses contact. Show that θ can be calculated by the help of the equation below

$$\cos\theta - (a/g)\sin\theta = 2/3$$

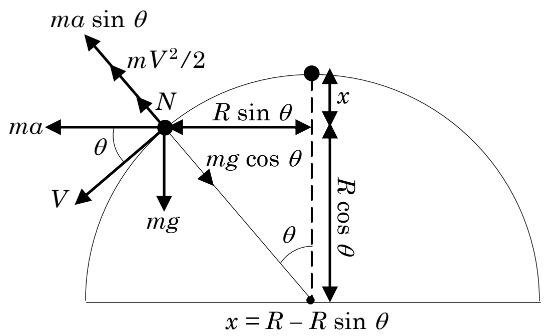

Diagram 9.10 Problem 9.10

Also show that the velocity of the body relative to the sphere at that instant will be equal to $[2gR/3]^{1/2}$.

Solution: In the frame of reference of the hemisphere only the force ma and mg can perform work. Refer to Diagram 9.10.
From energy balance,

$$mg\,(R - R\cos\theta) + maR\sin\theta = \frac{1}{2}mV^2 \qquad [i]$$

From force balance,

$$N + \frac{mV^2}{R} + ma\sin\theta = mg\cos\theta \qquad [ii]$$

Since $N = 0$, thus from Eqns. (i), and (ii),

$$V = \sqrt{2gR/3} \qquad [iii]$$

Again using Eqn. (ii), and (iii), $$\frac{m}{R}\frac{2}{3}gR + ma\sin\theta = mg\cos\theta$$

or, $$\cos\theta - (a/g)\sin\theta = 2/3$$

Problem 9.11: A particle [Diagram 9.11a] connected by a string of length l is given a horizontal velocity V. The string slackens at some point on the upper half of the circle. The particle moves on a parabola and passes through its initial position. Find the initial velocity given to the particle.

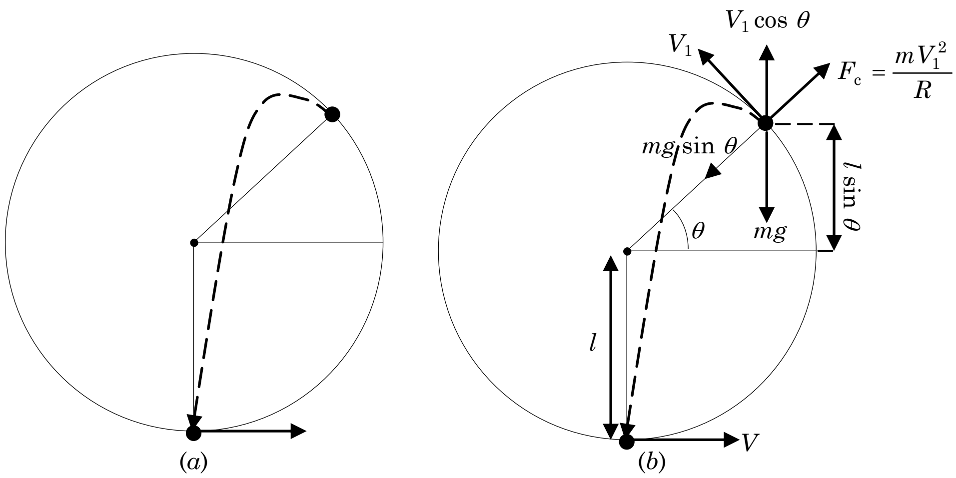

Diagram 9.11 Problem 9.11

Solution: Refer to Diagram 9.11b.

From energy balance, $$-mg\,(l + l\sin\theta) = \frac{1}{2}mV_1^2 - \frac{1}{2}mV^2$$

From force balance, $$mg\cos\theta = \frac{mV_1^2}{l}$$

Therefore, $$gl + \frac{3gl\sin\theta}{2} = \frac{V^2}{2}$$

or, $$V^2 = gl\,(2 + 3\sin\theta)$$

Problem 9.12: A small particle is placed inside circular surface, which is incomplete at the top, a shown in Diagram 9.12a. What should be the velocity of the particle at the lowest point so that it can join the circular surface after leaving it?

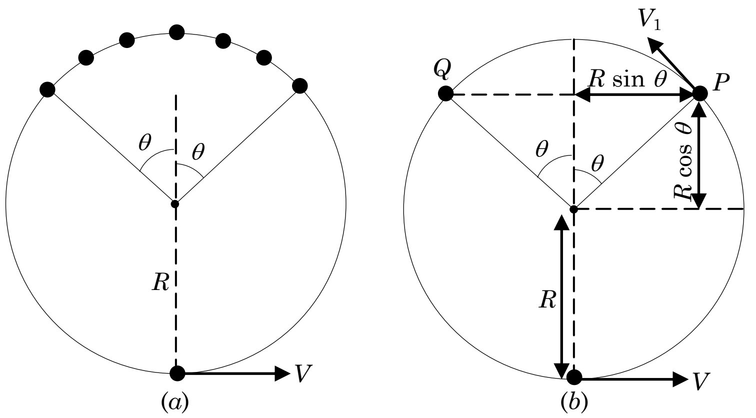

Diagram 9.12 Problem 9.12

Solution: At the point *P* the particle leaves the circle and rejoins it at point *Q* [Diagram 9.12b]. This is possible only if the normal reaction at the point *P* is non-zero.
From energy balance,

$$-mg\,(R + R\sin\theta) = \frac{1}{2}mV_1^2 - \frac{1}{2}mV^2 \qquad [i]$$

From force balance, $$2mg\sin\theta = \frac{mV_1^2}{R}\sin 2\theta$$

$$V_1^2 = \frac{gR}{\cos\theta} \qquad [ii]$$

From Eqn. (*i*), and (*ii*), $$V = \sqrt{gR\,(2 + 2\cos\theta + \sec\theta)}$$

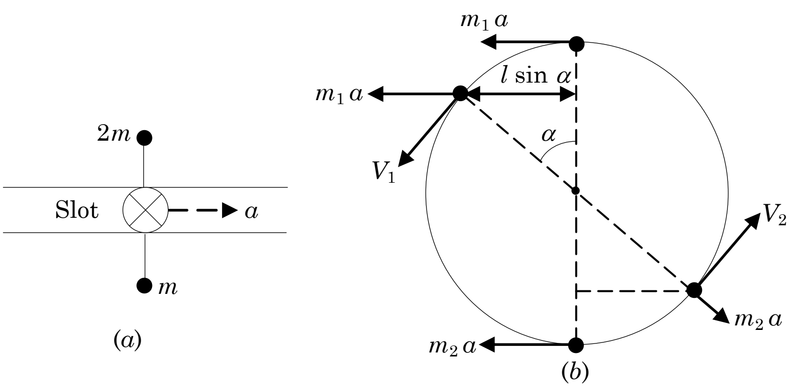

Diagram 9.13 Problem 9.13

Problem 9.13: Two point masses of *m* and 2*m* have been fixed at the ends of rigid rod which can rotate about the point *O* [Diagram 9.13a]. Assume gravity to be absent and the rod to be rigid and massless. The center *O* of the rod is constrained to move inside a slot. The center is being accelerated with a inside the horizontal slot. Find out the maximum angular velocity of the rod during the motion. Length of rod is l.

Solution: Refer to Diagram 9.13b, $m_1 = 2m$, $m_2 = m$, $l_1 = l_2$, $V_1 = \omega l$, $V_2 = \omega l$.
From energy balance,

$$m_1 al \sin\alpha - m_2 al \sin\alpha = \frac{1}{2}m_1V^2 - \frac{1}{2}m_2V^2$$

or,
$$2mal \sin\alpha - mal \sin\alpha = \frac{1}{2}2m(\omega l)^2 - \frac{1}{2}m(\omega l)^2$$

or,
$$\omega = \sqrt{\frac{2al\sin\alpha}{3l}}$$

Problem 9.14: In the previous question, let the rod have an angular displacement of θ. Find out the normal reaction acting at the center O as a function of θ. If the whole system is moving under constant acceleration A then find out the force applied by the external agent at the center O as function of θ.

Solution: Refer to Diagram 9.14.

$$a_{CM} = a - y_1\alpha.$$

and,
$$2mal - mal = I\alpha$$

or,
$$\alpha = a/l.$$

Thus,
$$y_1 = \frac{2ml - ml}{3m} = \frac{ml}{3}$$

and,
$$a_{CM} = a - \frac{l}{3}\frac{a}{l} = \frac{2a}{3}$$

Therefore,
$$F = 4m\frac{2a}{3} = \frac{8ma}{3}$$

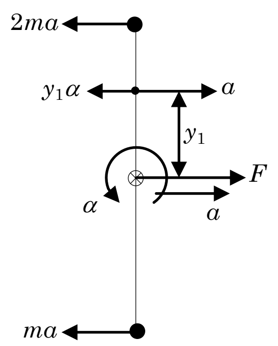

Diagram 9.14 Problem 9.14

Put the value of a from Problem 9.13, in above equation so we get the F as a function of θ.